Characterization of *Congruence in Sectionally * Semi Lattice

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Abstract: This paper illustrates several principal results concerning congruence kernels of sectionally pseudo complemented semilattices will also hold in Sectionally * semi lattice. Also it provides necessary and sufficient conditions such that any subset of sectionally*semi lattice which satisfies these conditions is kernel of some congruence. This paper institutes the notion of * ideals in Sectionally * semi lattice and demonstrates that every kernel ideal is a * ideal. As well it establishes a condition for smallest * congruence of Sectionally * semi lattice with kernel ideal.

Keywords: Semilattice, Pseudocomplemented semilattice, Sectionally*semilattice, congruence kernel, * Congruence, ideal.
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Introduction: W.H.Cornish[7] has investigated a congruence relation on pseudo complemented distributive lattices and has identified those ideals that are congruence kernels. T.S.Blyth [6] has showed results concerning congruence kernels and co kernels hold in semi -lattice and therefore do not depend on distributivity, nor on the existence of unions. By [4] several principal results concerning congruence kernels of pseudo complemented semilattices will also hold in sectionally pseudo complemented semilattices. Also it provides necessary and sufficient conditions such that any subset of sectionally pseudo complemented semilattice which satisfies these conditions is kernel of some congruence

In this paper we introduce the notion of * ideal in sectionally*semi lattice and proved that every kernel ideal is a * ideal. We also established a condition for smallest * congruence of sectionally pseudo complemented semilattice with kernel ideal.

1.1 Definition[6]: An element a ` is pseudo complement of an element a, if  a a = 0 and a\ x = 0  x\ a .

1.2. Theorem [6]: Let L be a pseudo complemented semilattice S (L) = {a / a є L} (be set of all pseudo complements). Then the partial ordering of L partially orders S (L) and makes S (L) into a Boolean lattice for a, b є S (L), we have a \ b є S (L) and join in S (L) is described by a V b = (a \ b) .

1.3. Result [6]: Let L be a pseudo complemented semilattice and let a, b є L. Then
(a\ b) = (a\ a\ b) = (a\ a\ b) .

1.4 Definition: A meet semi lattice S with 0 is said to be a * semi lattice if and only if for any a in S, there exists a` in S, such that (a`) = (a ) * , where a\ a = 0.

1.5 Note: A set A` stands for { x є S / x\ a = 0 for all a є A}, where A is non empty subset of S. If A = {a} then A = (a`).

1.6 Definition: A meet semi lattice S with 0 is said to be a sectionally * semi lattice if and only if for every a in S, the interval [0,a] is a * semi lattice.

1.7 Result [3]: Every Pseudo Complemented Semi lattice is sectionally *semi lattice.

1.8 Definition[7]: If (S, A, *, 0) is a sectionally *semilattice then by a * congruence (we shall mean) is a semilattice congruence ≡ that satisfies the additional condition x = y implies x* = y* for x, y є S.

1.9 Definition: Let (S, A, *, 0) be a sectionally*semilattice and R be a binary relation on S denoted by [0]R = { x є S; < x,0 >є R }. Especially, if R is a congruence on S, [0]R is called a congruence kernel (of R).

1.10 Theorem: If S is sectionally * semilattice, then a semilattice congruence ≡ on S is a * congruence if and only if x = 0,implies x* = 1
Proof: Let S be sectionally *semilattice, then for x, y є [0,1] є S, sectionally *semilattice is * congruence if x = y ⇒ x* = y* on S. Put y = 0 then x = 0 ⇒ x* = 1 in S. Thus condition is necessary.
Suppose, conversely that the condition hold and let \( x \equiv y \) for \( x, y \in S \). Then \( 0 = x \wedge x* \equiv y \wedge x* \Rightarrow x \wedge y \equiv 0 \). Thus \( (x \wedge y)* = 1 \).

Using the identity \( x \wedge (x \wedge y)* = x \wedge y* \) and since \( 1 \equiv x* \), we have
\[
1 = x* = x* \wedge x* \equiv x* \wedge 1 = x* \wedge y* \quad \text{by using identity}.
\]
Therefore \( x* \equiv x* \wedge y* \). Similarly \( y* \equiv x* \wedge y* \).

Hence \( x* \equiv y* \).

1.11. Definition: (Ideal) - A non-empty subset I of Sectionally * semilattice S is called an ideal if (i) for \( x, y \in I \), \( x \equiv y \equiv 1 \) and (ii) for \( x \in I, t \in S \) such that \( x \leq t \) implies \( t \in I \).

1.12. Definition: : An ideal I of sectionally *semilattice S will be called a kernel ideal if I is the kernel of a * congruence on S.

1.13. Theorem: An ideal I of sectionally *semilattice S is a kernel ideal of S if and only if \( i, j \in S \) implies \( (i* \wedge j*)* \equiv 1 \).

Proof: If I is kernel of a * congruence \( \equiv \) and if \( i, j \in I \), then \( I \equiv 0 \), implies \( i* \equiv 1 \);
Similarly \( j* \equiv 0 \) implies \( i* \equiv 1 \).

Therefore \( i* \equiv j* \equiv 1 \).

Hence \( i* \equiv j* \equiv 1 \).

Conversely, suppose that the condition holds, consider the relation “~” defined on S by
\[
x \sim y \iff \text{there exist } i \in I \text{ such that } x \wedge i* = y \wedge i* \iff x \sim x ,
\]
thus “~” is reflexive.

Let \( x \sim y \), then \( x \wedge i* = y \wedge i* \), implies \( y \wedge i* = x \wedge i* \), implies \( y \sim x \).
Therefore the relation “~” is symmetric.

Let \( x \sim y \) and \( y \sim z \), then \( x \wedge i* = y \wedge i* \) and \( y \wedge j* = z \wedge j* \) for \( i, j \in I \), as condition holds for \( i, j \in I \), (\( i* \wedge j*)* \equiv 1 \).

Let \( k = (i* \wedge j*)* \in I \). Now \( x \wedge k* = x \).

Hence the relation “~” is transitive.
It is clear that the equivalence relation “~” is sectionally *semilattice congruence on S.

Now by taking \( i=j \) in the condition, we obtain \( i \in I \), implies \( (i* \wedge i*)* = 1 \).

Thus we have \( x \sim 0 \iff \text{there exist } i \in I \), \( x \leq i* \text{ implies } x \equiv 1 \).

Therefore, if S is sectionally *semilattice then a semi-lattice congruence \( \equiv \) on S is a * congruence if and only if \( x \equiv 0 \) implies \( x* \equiv 1 \).

Hence the relation “~” is a * congruence on S.

1.14. Theorem: I is a kernel ideal if and only if (i) \( i \in I \Rightarrow i** \in I \) and (ii) for all \( i, j \in I \), there exist \( k \in I \) such that \( i* \wedge j* = k* \).

Proof: By above result, If I is kernel ideal then for \( i \in I \), implies \( i** \in I \) and for all \( i, j \in I \), there exists \( k \in I \) such that \( i* \wedge j* = k* \).

Conversely, suppose that the conditions (i) and (ii) holds then for all \( i, j \in I \), there exists \( k \in I \) such that \( i* \wedge j* = k* \), implies \( i* \wedge j* = k* \), then by condition (i) for \( k \in I, k** \in I \).

Therefore by above result, I is kernel.

1.15. Definition: An ideal I of sectionally *semilattice S will be called
* Ideal if \( i \in I \), implies \( i** \in I \).

1.16Example: as \( i* \) is the pseudo complement of \( i \) then \( i** = i \in I \).

1.17. Theorem: Every kernel ideal is a * ideal.

Proof: Let I be a kernel ideal of sectionally *semilattice S then by theorem 1.9, we have I is kernel ideal if and only if \( i \in I \), implies \( i** \in I \) and for all \( i, j \in I \) such that \( k* \in I \), implies \( i* \wedge j* = k* \). Thus I is a * ideal.

1.18. Theorem: Let S be sectionally *semilattice S and let I be a kernel ideal of S, then the smallest * congruence on S with kernel I is given by a relation \( R \) as \( x \sim y \) if and only if \( x \equiv I, x \wedge i* = y \wedge i* \).

Proof: If I is a kernel of a * congruence \( \equiv \) and if \( i, j \in I \), then \( i=0 \Rightarrow i* = a \).
Similarly \( j \equiv 0 \Rightarrow j^* \equiv a \) in \( S \) (i.e., in an interval \([0, a]\)). Therefore \( i^* \equiv j^* \equiv a \), then \( i^* \Lambda j^* \equiv a \).

Hence \( (i^* \Lambda j^*)^* \equiv 0 \in I \).

Conversely, suppose that the condition holds. Define a relation \( R \) on \( S \) as follows: \( x \ R \ y \) if and only if there exists \( i \in I \) such that \( x\Lambda i^* = y\Lambda i^* \).

Since \( x\Lambda i^* = x\Lambda i^* \) for \( i \in I \) if and only if \( x \ R \ x \), thus \( R \) is reflexive.

Let \( x \ R \ y \) then \( x\Lambda i^* = y\Lambda i^* \), \( \Rightarrow y\Lambda i^* = x\Lambda i^* \), \( \Rightarrow y \ R \ x \). Thus \( R \) is symmetric.

Let \( x \ R \ y \) and \( y \ R \ z \), then \( x\Lambda i^* = y\Lambda i^* \) and \( y\Lambda j^* = z\Lambda j^* \) for \( i, j \in I \).

As condition holds, for \( i, j \in I \), \( (i^* \Lambda j^*)^* \in I \). Let \( k = (i^* \Lambda j^*)^* \in I \).

Now \( x\Lambda k^* = x \Lambda (i^* \Lambda j^*)^* = x \Lambda i^* \Lambda j^* = y \Lambda i^* \Lambda j^* = z \Lambda i^* \Lambda j^* = z\Lambda k^* \). Thus \( x \ R \ z \). Hence \( R \) is transitive. It is clear that the equivalence relation \( R \) is semilattice congruence on \( S \).

Now by taking \( i = j \) in the condition obtains \( (i^* \Lambda j^*)^* = i^* \in I \). Thus we have \( x \ R \ 0 \) if and only if there exists \( i \in I \), \( x \Lambda i^* = 0 \Lambda i^* = 0 \) if and only if there exists \( i \in I \), \( x \leq i^* \) if and only if \( x \in I \), i.e., \( x \equiv 0 \). So the kernel of \( R \) is \( I \). Also \( x \ R \ a \) if and only if there exist \( i \in I, x\Lambda i^* = a \Lambda i^* = i^* \) if and only if there exists \( i \in I, x \geq i^* \), so that \( x \ R \ 0 \) implies there exists \( i \in I, x\Lambda i^* = 0 \), implies there exists \( i \in I \) such that \( i^* \leq x^* \Rightarrow x^* \ R \ a \), i.e., \( x^* \equiv a \).

We know that if \( S \) is sectionally*semilattice, then a semilattice congruence \( \equiv \) on \( S \) is a smallest * congruence if and only if \( x \equiv 0 \) implies \( x^* \equiv a \).

Hence the relation \( R \) is a * congruence on \( S \).

**Conclusion:**

This study illustrates many of principal results concerning congruence kernels hold in sectionally*semilattices. It is verified that every kernel ideal is a * ideal. Also it has been established a condition for smallest * congruence of sectionally pseudo complemented semilattice with kernel ideal.

**References:**